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$$\begin{cases} \frac{\partial}{\partial t} r\rho = D \frac{\partial^2}{\partial r^2} r\rho \\ \frac{\partial}{\partial t} rT = a \frac{\partial^2}{\partial r^2} rT \end{cases}$$

$\rho = \rho(r, t)$, $T = T(r, t)$

$\rho(r, 0) = \rho_\infty$, $T(r, 0) = T_\infty$

make Laplae transform:

$$P(r, s) = \int_0^\infty \rho(r, t) e^{-st} dt$$

$$\Theta(r, s) = \int_0^\infty T(r, t) e^{-st} dt$$

$$P(r, s) = \left[\frac{\rho e^{-st}}{-s} \right]_0^\infty - \int_0^\infty \frac{d\rho}{dt} \frac{e^{-st}}{-s} dt$$

$$= \frac{\rho_\infty}{s} + \frac{1}{s} \int_0^\infty \frac{d\rho}{dt} e^{-st} dt$$

$\because \rho(r, 0) = \rho_\infty$

$$\therefore \int_0^\infty \frac{d\rho}{dt} e^{-st} dt = sP(r, s) - \rho_\infty$$

similarly, $\int_0^\infty \frac{dT}{dt} e^{-st} dt = s\Theta(r, s) - T_\infty$

$$\therefore \begin{cases} \frac{d^2}{dr^2} rP = \frac{r}{D}[sP - \rho_\infty] \\ \frac{d^2}{dr^2} r\Theta = \frac{r}{a}[s\Theta - T_\infty] \end{cases}$$

Laplace method:

$$(a_n + b_n x) \frac{d^n y}{dx^n} + \cdots + (a_0 + b_0 x) y = 0$$

let $y = \int_C Z(t) e^{xt} dt$

$$\frac{d^p y}{dx^p} = \int_C Z(t) t^p e^{xt} dt$$

let $P(t) = a_n t^n + \cdots + a_0$

$Q(t) = b_n t^n + \cdots + b_0$

$$\therefore \int_c z(t) e^{xt} (P(t) + x Q(t)) dt = 0$$

$$0 = \int_c z(t) P(t) e^{xt} dt + \int_c z(t) Q(t) x e^{xt} dt$$

$$= \int_c z(t) P(t) e^{xt} dt + \left[z(t) Q(t) e^{xt} \right]_1^2 - \int_c \frac{d}{dt}\left(z(t) Q(t) \right) e^{xt} dt$$

$$= \int_c \left[z(t) P(t) - \frac{d}{dt}\left(z(t) Q(t) \right) \right] e^{xt} dt + \left[z(t) Q(t) e^{xt} \right]_1^2$$

if C is chosen so that $\left[z(t) Q(t) e^{xt} \right]_1^2 = 0$

then $\left[z(t) P(t) - \frac{d}{dt}\left(z(t) Q(t) \right) \right] = 0$

$$\frac{d}{dt} zQ = zP$$

$$\frac{d\, zQ}{zQ} = \frac{P}{Q} dt$$

$$\ln zQ = \int \frac{P}{Q} dt + \alpha'$$

$$\therefore z(t) = \frac{\alpha}{Q(t)} e^{\int \frac{P(t)}{Q(t)} dt}$$

$$rP'' + 2P' - brP = -\frac{r\, P_{00}}{D}, \quad b = \frac{s}{D}$$

homogeneous solution $\Rightarrow rP'' + 2P' - brP = 0$

$P(t) = 2t$

$Q(t) = t^2 - b$

$$\therefore z(t) = \frac{\alpha}{t^2 - b} e^{\int \frac{2t}{t^2 - b} dt} = \frac{\alpha}{t^2 - b} e^{\int \left(\frac{1}{t + b} + \frac{1}{t - b} \right) dt} = \alpha$$

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$z(t)\,Q(t)\,e^{rt} = \alpha(t^2 - b)\,e^{rt}$, choose $C = [-\sqrt{b}, \sqrt{b}]$

$$\therefore P = \int_{-\sqrt{b}}^{\sqrt{b}} \alpha e^{rt}\,dt = \frac{\alpha}{r}\left[e^{\sqrt{b}r} - e^{-\sqrt{b}r}\right]$$

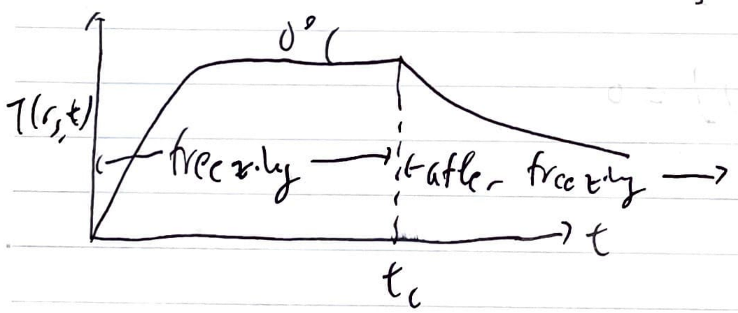
$r \to \infty$, should be finite, $\therefore P = -\frac{\alpha}{r} e^{-\sqrt{b}r} = \frac{A}{r} e^{-\sqrt{\frac{s}{D}}r}$

particular solution:

$$P_{ps} = \frac{P_\infty}{s}$$

$$\therefore P = \frac{A}{r} e^{-\sqrt{\frac{s}{D}}r} + \frac{P_\infty}{s}$$

similarly, $\theta = \frac{B}{r} e^{-\sqrt{\frac{s}{a}}r} + \frac{T_\infty}{s}$



① $T(r_s,t) = T_\infty + (T_0 - T_\infty)(1 - e^{-kt}) = T_\infty + (T_0 - T_\infty)G(t)$

② $P_{sat}(T) = P_\infty(1 + nT + mT^2) \Rightarrow P_{sat}(T) = P_\infty[1 + n(T_\infty + (T_0 - T_\infty)G(t)) + m(T_\infty^2 + 2T_\infty(T_0 - T_\infty)G(t) + (T_0 - T_\infty)^2 G^2(t))]$

$p(\infty, t) = P_\infty$, $T(\infty, t) = T_\infty$

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basic Laplace transform:

$$\int_0^\infty e^t e^{-st}\,dt = \frac{1}{1-s}\left[e^{(1-s)t}\right]_0^\infty = \frac{-1}{1-s} = \frac{1}{s-1}, \quad s > 1$$

$$\int_0^\infty e^{at} e^{-st}\,dt = \frac{1}{s-a}, \quad s > a$$

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Laplace transform of ① and ②

$$\theta(r_1, s) = \frac{T_\infty}{s} + (T_0 - T_\infty)\left(\frac{1}{s} - \frac{1}{s+K}\right)$$

$$p(r_1, s) = \frac{\rho_\infty}{s} + n\frac{\rho_\infty T_\infty}{s} + n\rho_\infty(T_0 - T_\infty)\left(\frac{1}{s} - \frac{1}{s+K}\right)$$

$$+ \frac{m\rho_\infty T_\infty^2}{s} + 2m\rho_\infty T_\infty(T_0 - T_\infty)\left(\frac{1}{s} - \frac{1}{s+K}\right)$$

$$+ m\rho_\infty(T_0 - T_\infty)^2\left(\frac{1}{s} - \frac{2}{s+K} + \frac{1}{s+2K}\right)$$

$$= \frac{\rho_\infty}{s} + n\rho_\infty T_0 \cdot \frac{1}{s} - n\rho_\infty(T_0 - T_\infty)\cdot\frac{1}{s+K}$$

$$+ m\rho_\infty\left[T_\infty(2T_0 - T_\infty) + (T_0 - T_\infty)^2\right]\frac{1}{s}$$

$$+ m\rho_\infty\left[-2T_\infty(T_0 - T_\infty) - 2(T_0 - T_\infty)^2\right]\frac{1}{s+K}$$

$$+ m\rho_\infty(T_0 - T_\infty)^2\frac{1}{s+2K}$$

$$= \frac{\rho_\infty}{s} + \frac{n\rho_\infty T_\infty}{s} + n\rho_\infty(T_0 - T_\infty)\frac{1}{s} - n\rho_\infty(T_0 - T_\infty)\frac{1}{s+K}$$

$$+ m\rho_\infty T_0^2\frac{1}{s} + m\rho_\infty(-2T_0^2)\frac{1}{s+K} + m\rho_\infty(T_0 - T_\infty)^2\frac{1}{s+2K}$$

$$= \frac{\rho_\infty}{s} + \left(n\rho_\infty T_0 + m\rho_\infty T_0^2\right)\frac{1}{s} - \left(n\rho_\infty(T_0 - T_\infty) + 2m\rho_\infty T_0^2\right)\frac{1}{s+K}$$

$$+ m\rho_\infty(T_0 - T_\infty)^2\frac{1}{s+2K}$$

$$= \frac{\rho_\infty}{s} + G_1\frac{1}{s} - G_2\frac{1}{s+K} + G_3\frac{1}{s+2K}$$

$$G_1 = \rho_\infty T_0(n + mT_0) \qquad G_3 = m\rho_\infty(T_0 - T_\infty)^2$$

$$G_2 = \rho_\infty\left(n(T_0 - T_\infty) + 2mT_0^2\right)$$

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subs them into the solutions to find A and B

$$\therefore A = \left[G_1 \frac{1}{s} - G_2 \frac{1}{s+k} + G_3 \frac{1}{s+2k} \right] r_s e^{\sqrt{\frac{s}{D}} r_s}$$

$$B = (T_0 - T_\infty)\left(\frac{1}{s} - \frac{1}{s+k}\right) r_s e^{\sqrt{\frac{s}{a}} r_s}$$

$$\therefore P(r,s) = \frac{P_\infty}{s} + \frac{r_s}{r}\left[G_1 \frac{1}{s} - G_2 \frac{1}{s+k} + G_3 \frac{1}{s+2k} \right] e^{-\sqrt{\frac{s}{D}}(r-r_s)}$$

$$\theta(r,s) = \frac{T_\infty}{s} + (T_0 - T_\infty)\frac{r_s}{r}\left(\frac{1}{s} - \frac{1}{s+k}\right) e^{-\sqrt{\frac{s}{a}}(r-r_s)}$$

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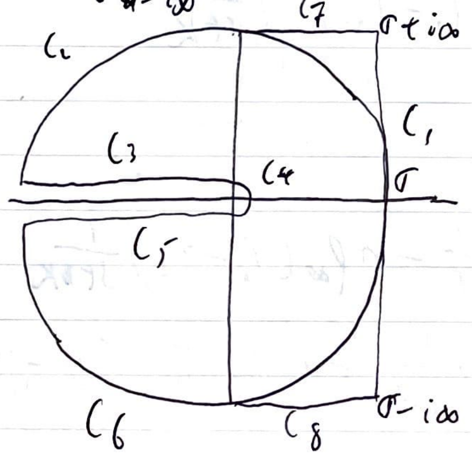
advanced Laplace transform:

$$\int_{\sigma - i\infty}^{\sigma + i\infty} e^{-\sqrt{as}} e^{st} \, ds = ? \quad , \; a \text{ is real}$$



let branch cut of $\sqrt{s}$ be negative axis

$$\therefore \oint e^{-\sqrt{as}} e^{st} \, ds = 0$$

C_2 and C_6

let $s = Re^{i\theta}$

$$e^{-\sqrt{aR} e^{i\frac{\theta}{2}}} e^{Rte^{i\theta}}$$

$$\mathbb{R}\left[e^{-\sqrt{aR} e^{i\frac{\theta}{2}}} e^{Rte^{i\theta}} \right] = e^{Rt\cos\theta - \sqrt{aR}\cos\frac{\theta}{2}}$$

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$\because$ for C_2, $\cos\theta < 0$ $\left(\frac{\pi}{2} \le \theta < \pi\right)$, $\cos\frac{\theta}{2} \ge 0$

for C_6, $\cos\theta < 0$ $\left(-\pi < \theta \le -\frac{\pi}{2}\right)$, $\cos\frac{\theta}{2} \ge 0$

$\therefore$ as $R \to \infty$, $\int_{C_2+C_6} e^{-\sqrt{as}} e^{st}\, ds = 0$

Same argument can be applied to C_7 and C_8

C_4

$s = \varepsilon e^{i\theta}$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\therefore \int_{C_4} e^{-\sqrt{a\varepsilon} e^{i\frac{\theta}{2}}} e^{\varepsilon t e^{i\theta}}\, d\theta \to 0$

C_3

$s = e^{i\pi} x \quad ds = e^{i\pi} dx$

let $x = -X$, $dx = -dX$

let $x = u^2$, $dx = 2u\,du$

$$\int_{\infty}^{0} e^{-i\sqrt{a}\sqrt{x}} e^{-xt}\, dx = \int_{\infty}^{0} e^{-xt - i\sqrt{ax}}\, dx$$

$$= \int_{\infty}^{0} e^{-u^2 t} e^{-i\sqrt{a}u} \cdot 2u\,du$$

$$= \int_{\infty}^{0} e^{-t\left(u + \frac{i\sqrt{a}}{2t}\right)^2 - \frac{a}{4t}} \cdot 2u\,du$$

$$= \int_{\infty}^{\frac{i\sqrt{a}}{2t}} e^{-tv^2} e^{-\frac{a}{4t}}\, 2\left(v - \frac{i\sqrt{a}}{2t}\right) dv$$

$$= \int_{\infty}^{\frac{i\sqrt{a}}{2t}} e^{-tv^2} e^{-\frac{a}{4t}} \cdot 2v\,dv + I_2$$

$$= \int_{\infty}^{-\frac{a}{4t^2}} e^{-ty}\, dy \cdot e^{-\frac{a}{4t}} + I_2$$

$$= \frac{-1}{t}\left[e^{\frac{a}{4t}} - 0\right] e^{-\frac{a}{4t}} + I_2$$

$$= -\frac{1}{t} + I_2$$

$$I_2 = -\int_{-\infty}^{0} e^{xt + \sqrt{a}x}\, dx$$

$$= \int_{0}^{\infty} e^{xt + \sqrt{a}x}\, dx$$

$$= \int_{0}^{\infty} e^{u^2 t + \sqrt{a}u} \cdot 2u\,du$$

$$= 2\int_{0}^{\infty} e^{t\left(u + \frac{\sqrt{a}}{2t}\right)^2 - \frac{a}{4t}} u\,du$$

let $v = u + \frac{\sqrt{a}}{2t}$, $dv = du$

$$= 2e^{-\frac{a}{4t}} \int_{\frac{\sqrt{a}}{2t}}^{\infty} e^{tv^2} v\,dv$$

let $y = v^2$, $dy = 2v\,dv$

$$= e^{-\frac{a}{4t}} \int_{\frac{a}{4t^2}}^{\infty} e^{ty}\, dy$$

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C_5

$s = e^{-i\pi} x \qquad ds = e^{-i\pi} dx$

$\int_0^\infty e^{i\sqrt{a}x} e^{-xt}\, dx$, $\because \sqrt{s} = e^{-\frac{i\pi}{2}} \sqrt{x} = -i\sqrt{x}$

$s = e^{-i\pi} x = -x$

$C_3 + C_5$

$e^{i\pi} \int_\infty^0 e^{-i\sqrt{a}x} e^{-xt} + e^{-i\pi} \int_0^\infty e^{i\sqrt{a}x} e^{-xt}\, dx$

$= -\left[\int_0^\infty (e^{i\sqrt{a}x} - e^{-i\sqrt{a}x}) e^{-xt}\, dx \right]$ let $x = u^2$

$= -\int_0^\infty (e^{i\sqrt{a}u} - e^{-i\sqrt{a}u}) e^{-u^2 t}\, 2u\,du$

$= -\frac{1}{2}\int_{-\infty}^\infty (e^{i\sqrt{a}u} - e^{-\sqrt{a}u}) e^{-u^2 t}\, 2u\,du$, even function

$C_1 + C_3 + C_5$

$\int_{\sigma - i\infty}^{\sigma + i\infty} e^{-\sqrt{a}s} e^{st}\, ds = \frac{1}{2}\int_{-\infty}^\infty (e^{i\sqrt{a}u} - e^{-i\sqrt{a}u}) e^{-u^2 t}\, 2u\,du$

$= i \int_{-\infty}^\infty \sin i\sqrt{a}u\, e^{-u^2 t} \cdot 2u\,du$

$= i\,\mathrm{I}\left[\int_{-\infty}^\infty e^{-u^2 t} e^{i\sqrt{a}u}\, 2u\,du \right]$

$= i\,\mathrm{I}\left[\int_{-\infty}^\infty e^{-t(u - \frac{i\sqrt{a}}{2t})^2 - \frac{a}{4t}}\, 2u\,du \right]$

$= i\,\mathrm{I}\left[\int_{-\infty}^\infty e^{-tv^2} e^{-\frac{a}{4t}}\, 2(v + \frac{i\sqrt{a}}{2t})\, dv \right]$

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↙ real

$$= i\,\mathrm{Im}\left[\int_{-\infty}^{\infty} e^{-tu^2} e^{-\frac{a}{4t}} \cdot 2u\,du + \int_{-\infty}^{\infty} e^{-tu^2} e^{-\frac{a}{4t}} \cdot \frac{i\sqrt{a}}{t}\,du\right]$$

$$= i\,\mathrm{Im}\left[e^{-\frac{a}{4t}} \cdot i\sqrt{\frac{a\pi}{t}}\frac{1}{t}\right]$$

$$= i\frac{\sqrt{a\pi}}{t^{3/2}} e^{-\frac{a}{4t}}$$

$$L^{-1}\left[e^{-\sqrt{as}}\right] = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{-\sqrt{as}} e^{st}\,ds$$

$$= \frac{1}{2}\sqrt{\frac{a}{\pi}}\frac{1}{t^{3/2}} e^{-\frac{a}{4t}}$$

$$L^{-1}\left[e^{-\sqrt{\frac{s}{a}}(r-r_s)}\right] = L^{-1}\left[e^{-\sqrt{\frac{(r-r_s)^2 s}{a}}}\right] = \frac{1}{2}\frac{r-r_s}{\sqrt{a\pi}}\frac{1}{t^{3/2}} e^{-\frac{(r-r_s)^2}{at}}$$

$$L^{-1}\left[\frac{1}{s}\right] = H(t)$$

$$L^{-1}\left[\frac{1}{s+k}\right] = e^{-kt}$$

$$L^{-1}\left[\frac{1}{s+2k}\right] = e^{-2kt}$$

$$L^{-1}\left[\frac{1}{s}e^{-\sqrt{\frac{s}{a}}(r-r_s)}\right] = \int_0^t \frac{1}{2}\frac{r-r_s}{\sqrt{a\pi}}\frac{1}{y^{3/2}} e^{-\frac{(r-r_s)^2}{ay}}\,dy$$

$$= -\frac{r-r_s}{\sqrt{a\pi}}\int_{\infty}^{\frac{1}{\sqrt{t}}} e^{-\frac{(r-r_s)^2}{a}u^2}\,du$$

$$= \frac{-1}{\sqrt{\pi}}\int_{\infty}^{\frac{r-r_s}{\sqrt{at}}} e^{-v^2}\,dv$$

$$= \frac{1}{\sqrt{\pi}}\int_{\frac{r-r_s}{\sqrt{at}}}^{\infty} e^{-v^2}\,dv$$

$$= \frac{1}{2}\,\mathrm{erfc}\left(\frac{r-r_s}{\sqrt{at}}\right)$$

$u = \frac{1}{\sqrt{y}}$, $y = \frac{1}{u^2}$

$du = \frac{-1}{2}\frac{1}{y^{3/2}}\,dy$

$u = \frac{\sqrt{a}}{r-r_s}v$

$du = \frac{\sqrt{a}}{r-r_s}\,dv$

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$$L^{-1}\left[\frac{1}{s+k} e^{-\sqrt{\frac{s}{a}}(r-r_s)}\right] = \int_0^t e^{-k(t-y)} \frac{1}{2}\frac{r-r_s}{\sqrt{a\pi}} \frac{1}{y^{3/2}} e^{-\frac{(r-r_s)^2}{ay}} dy$$

$$= e^{-kt} \cdot \frac{1}{2}\frac{r-r_s}{\sqrt{a\pi}} \int_0^t \frac{1}{y^{3/2}} e^{-\frac{(r-r_s)^2}{ay}} e^{ky} dy$$

$$= e^{-kt} \cdot \frac{1}{2}\frac{r-r_s}{\sqrt{a\pi}} \int_0^{\frac{1}{\sqrt{t}}} -e^{-\frac{(r-r_s)^2}{a}u^2 + \frac{k}{u^2}} du$$

$$\int_0^T e^{-a^2x^2-\frac{b^2}{x^2}} dx = I$$

$$I = \int_0^T e^{-(ax+\frac{b}{x})^2+2ab} dx = \int_0^T e^{-(ax-\frac{b}{x})^2-2ab} dx$$

let $u = \sqrt{\frac{a}{b}}x$, $du = \sqrt{\frac{a}{b}}dx$

$$\therefore I = \int_0^{\sqrt{\frac{a}{b}}T} e^{-ab(u+\frac{1}{u})^2+2ab} \sqrt{\frac{b}{a}} du = \int_0^{\sqrt{\frac{a}{b}}T} e^{-ab(u-\frac{1}{u})^2-2ab} \sqrt{\frac{b}{a}} du$$

$$2\sqrt{\frac{a}{b}} I = e^{2ab}\int_0^{\sqrt{\frac{a}{b}}T} e^{-ab(u+\frac{1}{u})^2} du + e^{-2ab}\int_0^{\sqrt{\frac{a}{b}}T} e^{-ab(u-\frac{1}{u})^2} du$$

$$= e^{2ab}\int_0^{\sqrt{\frac{a}{b}}T} (1-\frac{1}{u^2}) e^{-ab(u+\frac{1}{u})^2} du + e^{-2ab}\int_0^{\sqrt{\frac{a}{b}}T} (1+\frac{1}{u^2}) e^{-ab(u-\frac{1}{u})^2} du$$

$$+ \int_0^{\sqrt{\frac{a}{b}}T} \frac{1}{u^2} e^{-ab(u^2+\frac{1}{u^2})} du - \int_0^{\sqrt{\frac{a}{b}}T} \frac{1}{u^2} e^{-ab(u^2+\frac{1}{u^2})} du$$

let $v = u + \frac{1}{u}$, $w = u - \frac{1}{u}$

$dv = (1-\frac{1}{u^2})du$ $dw = (1+\frac{1}{u^2})du$

$$= e^{2ab}\int_{\infty}^{\sqrt{\frac{a}{b}}T+\sqrt{\frac{b}{a}}\frac{1}{T}} e^{-abv^2} dv + e^{-2ab}\int_{-\infty}^{\sqrt{\frac{a}{b}}T-\sqrt{\frac{b}{a}}\frac{1}{T}} e^{-abw^2} dw$$

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let $v = \frac{y}{\sqrt{ab}}$ $\quad dv = \frac{dy}{\sqrt{ab}}$

$$= \frac{1}{\sqrt{ab}} e^{2ab} \int_{\infty}^{ab(T+\frac{1}{T})} e^{-y^2} dy + \frac{1}{\sqrt{ab}} e^{-2ab} \int_{-\infty}^{ab(T-\frac{1}{T})} e^{-y^2} dy$$

or

$$I = \frac{e^{2ab}}{a} \int_0^T \left(a - \frac{b}{x^2}\right) e^{-(ax+\frac{b}{x})^2} dx + \frac{e^{-2ab}}{a} \int_0^T \left(a + \frac{b}{x^2}\right) e^{-(ax-\frac{b}{x})^2} dx$$

$$= \frac{e^{2ab}}{a} \int_{\infty}^{aT+\frac{b}{T}} e^{-u^2} du + \frac{e^{-2ab}}{a} \int_{-\infty}^{aT-\frac{b}{T}} e^{-u^2} du$$

$$= \frac{e^{2ab}}{a} \cdot \frac{-\sqrt{\pi}}{2} + \frac{e^{2ab}}{a} \int_0^{aT+\frac{b}{T}} e^{-u^2} du + \frac{e^{-2ab}}{a} \cdot \frac{\sqrt{\pi}}{2} + \frac{e^{-2ab}}{a} \int_0^{aT-\frac{b}{T}} e^{-u^2} du$$

$$= \frac{\sqrt{\pi}}{2a} \left[e^{-2ab} - e^{2ab} + e^{2ab} \operatorname{erf}\left(aT+\frac{b}{T}\right) + e^{-2ab} \operatorname{erf}\left(aT-\frac{b}{T}\right) \right]$$

$$\therefore e^{2ab} \operatorname{erf}\left(aT+\frac{b}{T}\right) + e^{-2ab}\left(aT-\frac{b}{T}\right) = \frac{4a}{\sqrt{\pi}} I + e^{2ab} - e^{-2ab}$$

$$\therefore I = \frac{-\sqrt{\pi}}{4a} \left[e^{2ab} \operatorname{erfc}\left(aT+\frac{b}{T}\right) + e^{-2ab} \operatorname{erfc}\left(aT-\frac{b}{T}\right) \right]$$

alternatively, $$I = \frac{-\sqrt{\pi}}{4a} \left[e^{2ab} \operatorname{erfc}\left(ab\left(T+\frac{1}{T}\right)\right) + e^{-2ab} \operatorname{erfc}\left(ab\left(T+\frac{1}{T}\right)\right) \right]$$

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Laplace transform of $e^{at} \operatorname{erf}(\sqrt{at})$

$$\frac{2}{\sqrt{\pi}} \int_0^\infty e^{at} \int_0^{\sqrt{at}} e^{-u^2} du \, e^{-st} dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^{\sqrt{at}} e^{-u^2} du \cdot e^{(a-s)t} dt$$

$$= \frac{2}{\sqrt{\pi}} \left[\frac{1}{a-s} \left[\int_0^{\sqrt{at}} e^{-u^2} du \cdot e^{(a-s)t} \right]_0^\infty - \frac{1}{a-s} \int_0^\infty e^{-at} \frac{1}{2} \sqrt{\frac{a}{t}} e^{(a-s)t} dt \right]$$

$$= - \sqrt{\frac{a}{\pi}} \frac{1}{a-s} \int_0^\infty t^{-\frac{1}{2}} e^{-st} dt \qquad u = ts, \; du = s\,dt$$

$$= - \sqrt{\frac{a}{\pi}} \frac{1}{a-s} \int_0^\infty \sqrt{\frac{s}{u}} e^{-u} \frac{du}{s}$$

$$= \sqrt{\frac{a}{\pi}} \frac{1}{\sqrt{s}(s-a)} \int_0^\infty u^{-\frac{1}{2}} e^{-u} du$$

$$= \frac{\sqrt{a}}{\sqrt{s}(s-a)}$$

Laplace transform of $e^{a^2 x} \operatorname{erfc}(a\sqrt{x})$

$$\operatorname{erfc}(a\sqrt{x}) = \frac{2}{\sqrt{\pi}} \int_{a\sqrt{x}}^\infty e^{-u^2} du$$

$$\int_0^\infty e^{a^2 t} \cdot \frac{2}{\sqrt{\pi}} \int_{a\sqrt{t}}^\infty e^{-u^2} du \, e^{-st} dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty \int_\infty^{a\sqrt{t}} -e^{-u^2} du \cdot e^{(a^2-s)t} dt$$

$$= \frac{2}{\sqrt{\pi}} \left[\left[\frac{1}{a^2-s} \int_\infty^{a\sqrt{t}} -e^{-u^2} du \cdot e^{(a^2-s)t} \right]_0^\infty + \frac{1}{a^2-s} \int_0^\infty e^{-a^2 t} \cdot \frac{a}{2\sqrt{t}} \cdot e^{(a^2-s)t} dt \right]$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{a^2-s} \int_\infty^0 e^{-u^2} du + \frac{a}{\sqrt{\pi}} \frac{1}{a^2-s} \int_0^\infty t^{-\frac{1}{2}} e^{-st} dt$$

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$$= \frac{1}{s-a^2} + \frac{a}{s-a^2}\frac{1}{\sqrt{s}}$$

$$= \frac{1}{(\sqrt{s}+a)(\sqrt{s}-a)}\left(\frac{\sqrt{s}+a}{\sqrt{s}}\right)$$

$$= \frac{1}{s+a\sqrt{s}} \#$$

also, $L(1) - L(e^{a^2 t} \operatorname{erfc}(a\sqrt{t})) = L(1 - e^{a^2 t}\operatorname{erfc}(a\sqrt{t}))$

$$= \frac{1}{s} - \frac{1}{s+a\sqrt{s}}$$

$$= \frac{a\sqrt{s}}{s(s+a\sqrt{s})} = \frac{a}{s(\sqrt{s}+a)}$$

Laplace transform of $e^{-ax}\operatorname{erf}(\sqrt{(b-a)x})$ and

$$\frac{e^{-ax}}{\sqrt{\pi x}} + \sqrt{a-b}\, e^{-bx}\operatorname{erf}(\sqrt{(a-b)x})$$

$$L(e^{-at}\operatorname{erf}(\sqrt{(b-a)x}))$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty \int_0^{\sqrt{(b-a)t}} e^{-u^2}du \cdot e^{-(a+s)t}\,dt$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{-1}{a+s}\left[\int_0^{\sqrt{(b-a)t}} e^{-u^2}du \cdot e^{-(a+s)t}\right]_0^\infty + \frac{1}{a+s}\int_0^\infty \frac{1}{2}\sqrt{\frac{b-a}{t}}\, e^{-(b-a)t} \cdot e^{-(a+s)t}\,dt\right.$$

$$= \frac{1}{\sqrt{\pi}}\frac{\sqrt{b-a}}{a+s}\int_0^\infty t^{-\frac{1}{2}} e^{-(b+s)t}\,dt \qquad u = (b+s)t$$

$$du = (b+s)dt$$

$$= \frac{1}{\sqrt{\pi}}\frac{\sqrt{b-a}}{a+s}\frac{1}{\sqrt{b+s}}\int_0^\infty u^{-\frac{1}{2}}e^{-u}\,du$$

$$= \frac{\sqrt{b-a}}{(a+s)\sqrt{b+s}}$$

No.

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$$L\left(\frac{e^{-at}}{\sqrt{\pi t}}\right) = \int_0^\infty \frac{1}{\sqrt{\pi}} t^{-\frac{1}{2}} e^{-(a+s)t} dt$$

$u = (a+s)t$
$du = (a+s)dt$

$$= \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{a+s}} \int_0^\infty u^{-\frac{1}{2}} e^{-u} du$$

$$= \frac{1}{\sqrt{a+s}}$$

$$\therefore L\left(\frac{e^{-at}}{\sqrt{\pi t}} + \sqrt{a-b}\, e^{-bx} \operatorname{erf}\left(\sqrt{(a-b)x}\right)\right)$$

$$= \frac{1}{\sqrt{a+s}} + \frac{a-b}{(b+s)\sqrt{a+s}}$$

$$= \frac{1}{\sqrt{a+s}} \cdot \frac{a+s}{b+s} = \frac{\sqrt{a+s}}{b+s}$$

Laplace tranform of $2\sqrt{\frac{x}{\pi}}$

$$L\left(2\sqrt{\frac{t}{\pi}}\right) = \int_0^\infty \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} e^{-st} dt$$

$u = st$
$du = s\,dt$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty \sqrt{\frac{u}{s}} e^{-u} \frac{du}{s}$$

$$= \frac{1}{s\sqrt{s}} \frac{2}{\sqrt{\pi}} \int_0^\infty u^{\frac{1}{2}} e^{-u} du$$

$$= \frac{1}{s\sqrt{s}}$$

$$L^{-1}\left(\frac{1}{s} \frac{1}{a+\sqrt{s}} e^{-\sqrt{bs}}\right)$$

$$L^{-1}\left(\frac{1}{s} \frac{1}{a+\sqrt{s}}\right) = \frac{1}{\sqrt{a}} \operatorname{erf}(\sqrt{ax})$$

$$L^{-1}\left(e^{-\sqrt{bs}}\right) = \frac{1}{2}\sqrt{\frac{b}{\pi}} \frac{1}{t^{3/2}} e^{-\frac{b}{4t}}$$

$$\therefore L^{-1}\left(\frac{1}{s}\frac{1}{a+\sqrt{s}}e^{-\sqrt{bs}}\right)$$

$$= \int_0^T \frac{\sqrt{b}}{2a}\int_0^{\sqrt{a}t} e^{-u^2}\,du \cdot \frac{1}{(T-t)^{3/2}} e^{-\frac{b}{4(T-t)}}\,dt$$